جامعة بنها

كلية الحاسبات والذكاء الإصطناعي

Probability and Statistics

Lecture 04

Dr. Ahmed Hagag

Faculty of Computers and Artificial Intelligence
Benha University

Spring 2023

Chapter 2: Random Variable

- Discrete Random Variables.
- Probability Mass Functions.
- Cumulative Distribution Functions.
- Discrete R.V. (Mean and Variance).
- Continuous Random Variables.
- Probability Density Functions.
- Continuous R.V. (Mean and Variance).
- Joint Probability Distributions.

Discrete Random Variables (1/3)

Random Variable

- Is a function that assigns a real number to each outcome in the sample space of random experiment. Denoted by an uppercase letter such as X

A Discrete Random Variable

- Is a random variable with a finite (or countable infinite) range.
- The possible values of X may be listed as $x_1, x_2, \ldots$

Discrete Random Variables (2/3)

Example1

- Flipping a coin of two times. Let X is the number of heads.

Discrete Random Variables (3/3)

Example1

- Flipping a coin of two times. Let X is the number of heads.

Answer:

$$S = \{HH, HT, TH, TT\}$$

2 1 1 0

$$x = 0, 1, 2$$

$$P(0) = \frac{1}{4}, \quad P(1) = \frac{2}{4}, \quad P(2) = \frac{1}{4}$$

Probability Mass Fun. (1/14)

Probability Mass Function

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, a **probability mass function** is a function such that

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$

x_i	x_1	x_2	x_3	x_4	x_5
$f(x_i) = P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	$P(x_4)$	$P(x_5)$

Probability Mass Fun. (2/14)

Example1

Verify that the function is a probability mass function:

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Probability Mass Fun. (3/14)

Example1

Verify that the function is a probability mass function:

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer:

$\sum P(x_i) = 1$, $P(x_i) \geq 0$

Probability Mass Fun. (4/14)

Example2

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

a. $P(X \leq 2)$

b. $P(X > -2)$

c. $P(-1 \leq X \leq 1)$

d. $P(X \leq -1 \text{ or } X = 2)$

Probability Mass Fun. (5/14)

Example2

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer:

a. $P(X \leq 2) = 1$

b. $P(X > -2) = \frac{7}{8}$

c. $P(-1 \leq X \leq 1) = \frac{6}{8}$

d. $P(X \leq -1 \;\text{ or }\; X = 2) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8}$

Probability Mass Fun. (6/14)

Example3

Two balls are drawn in succession without replacement from a box containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y , where y is the number of red balls, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

No Red Balls

$$f(0) = P(Y = 0) = \frac{\binom{4}{0}\binom{3}{2}}{\binom{7}{2}} = \frac{3}{21} = \frac{1}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

One Red Ball (RB, BR)

$$f(1) = P(Y = 1) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{12}{21} = \frac{4}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Two Red Balls

$$f(2) = P(Y = 2) = \frac{\binom{4}{2}\binom{3}{0}}{\binom{7}{2}} = \frac{6}{21} = \frac{2}{7}$$

Probability Mass Fun. (7/14)

Example3

Sample Space	y
RR	2
RB	1
BR	1
BB	0

y	0	1	2
$f(y) = P(Y = y)$	1/7	4/7	2/7

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Note: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school.

Probability Mass Fun. (8/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Note: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. **Then x can only take the numbers 0, 1, and 2.**

Probability Mass Fun. (9/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{136}{190}$$

Probability Mass Fun. (10/14)

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$$

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$$

Example4

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

x	0	1	2
$f(x) = P(X = x)$	136/190	51/190	3/190

Probability Mass Fun. (13/14)

Example5

There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0, 1, 2, 3, 4\}$.

Suppose that the probabilities are

$$P(X=0)=0.6561 \quad P(X=1)=0.2916$$
$$P(X=2)=0.0486 \quad P(X=3)=0.0036$$
$$P(X=4)=0.0001$$

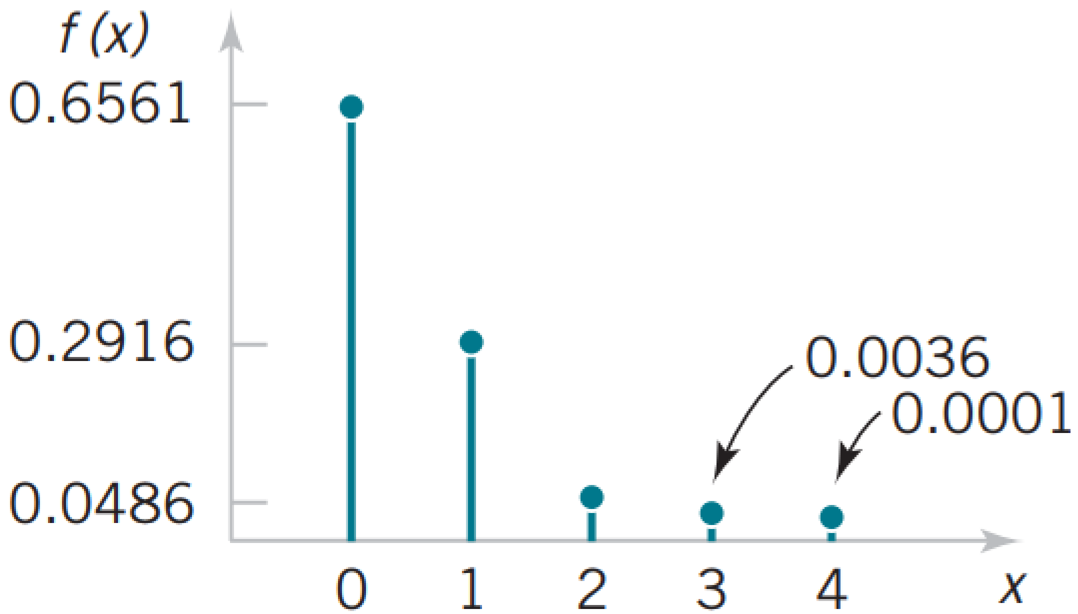
Probability Mass Fun. (14/14)

Example5

$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$

$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$

$P(X = 4) = 0.0001$



Cumulative Distribution (1/9)

The cumulative distribution function (cdf), denoted by $F(x)$, measures the probability that the random variable X assumes a value less than or equal to x, that is,

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

Cumulative Distribution (2/9)

If X is discrete, then

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Cumulative Distribution (3/9)

If X is discrete, then

$$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$$

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8
$\boldsymbol{F(x) = P(X \le x)}$	**1/8**	**3/8**	**5/8**	**7/8**	**8/8**

Cumulative Distribution (4/9)

Example1

$$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$$
$$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$$
$$P(X = 4) = 0.0001$$

x	0	1	2	3	4
$f(x) = P(X = x)$	0.6561	0.2916	0.0486	0.0036	0.0001

Cumulative Distribution (5/9)

Example1

$P(X = 0) = 0.6561 \quad P(X = 1) = 0.2916$

$P(X = 2) = 0.0486 \quad P(X = 3) = 0.0036$

$P(X = 4) = 0.0001$

x	0	1	2	3	4
$f(x) = P(X = x)$	0.6561	0.2916	0.0486	0.0036	0.0001
$\boldsymbol{F(x) = P(X \leq x)}$	**0.6561**	**0.9477**	**0.9963**	**0.9999**	**1**

Cumulative Distribution (6/9)

Example1

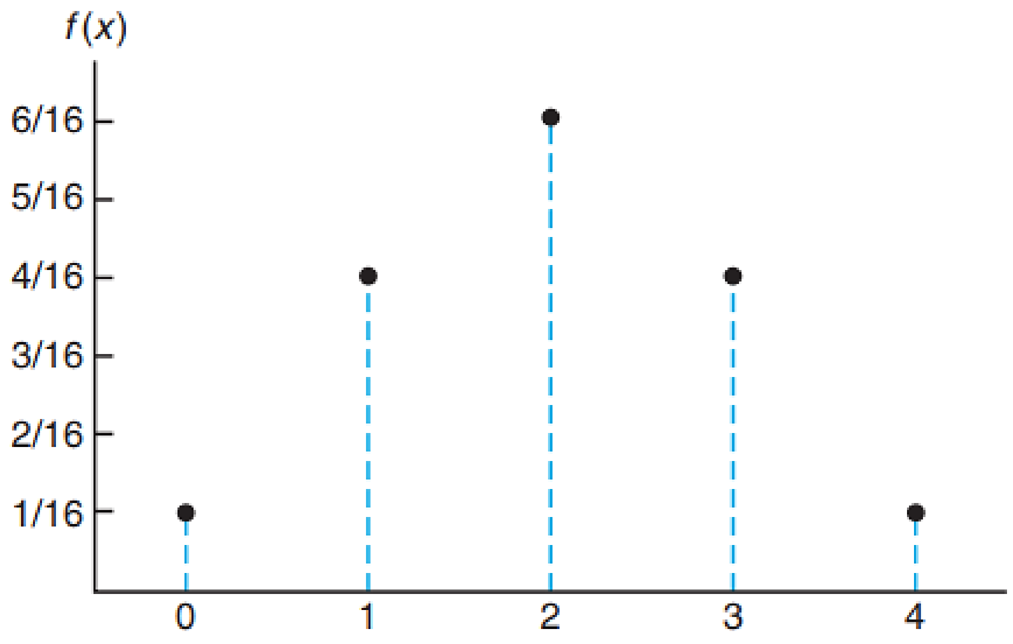
x	0	1	2	3	4
$f(x) = P(X = x)$	0.6561	0.2916	0.0486	0.0036	0.0001
$\boldsymbol{F(x) = P(X \leq x)}$	**0.6561**	**0.9477**	**0.9963**	**0.9999**	**1**

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.6561 & 0 \leq x < 1 \\ 0.9477 & 1 \leq x < 2 \\ 0.9963 & 2 \leq x < 3 \\ 0.9999 & 3 \leq x < 4 \\ 1 & 4 \leq x \end{cases}$$

Cumulative Distribution (7/9)

Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16



Probability mass function plot.

Cumulative Distribution (8/9)

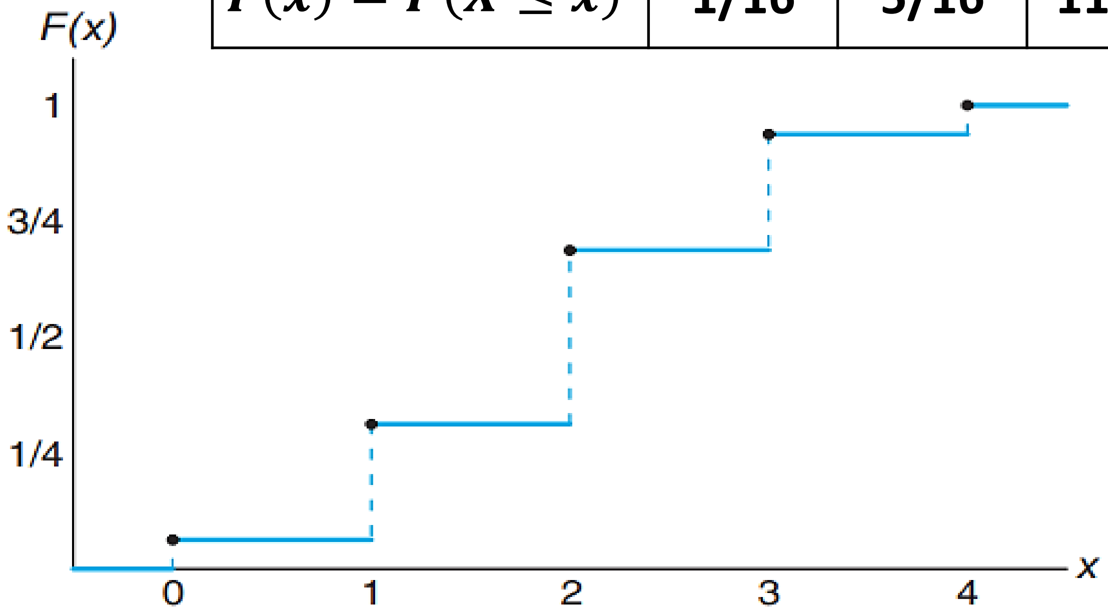
Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16
$\boldsymbol{F(x) = P(X \leq x)}$	**1/16**	**5/16**	**11/16**	**15/16**	**16/16**

Cumulative Distribution (9/9)

Example2

x	0	1	2	3	4
$f(x) = P(X = x)$	1/16	4/16	6/16	4/16	1/16
$\boldsymbol{F(x) = P(X \leq x)}$	**1/16**	**5/16**	**11/16**	**15/16**	**16/16**



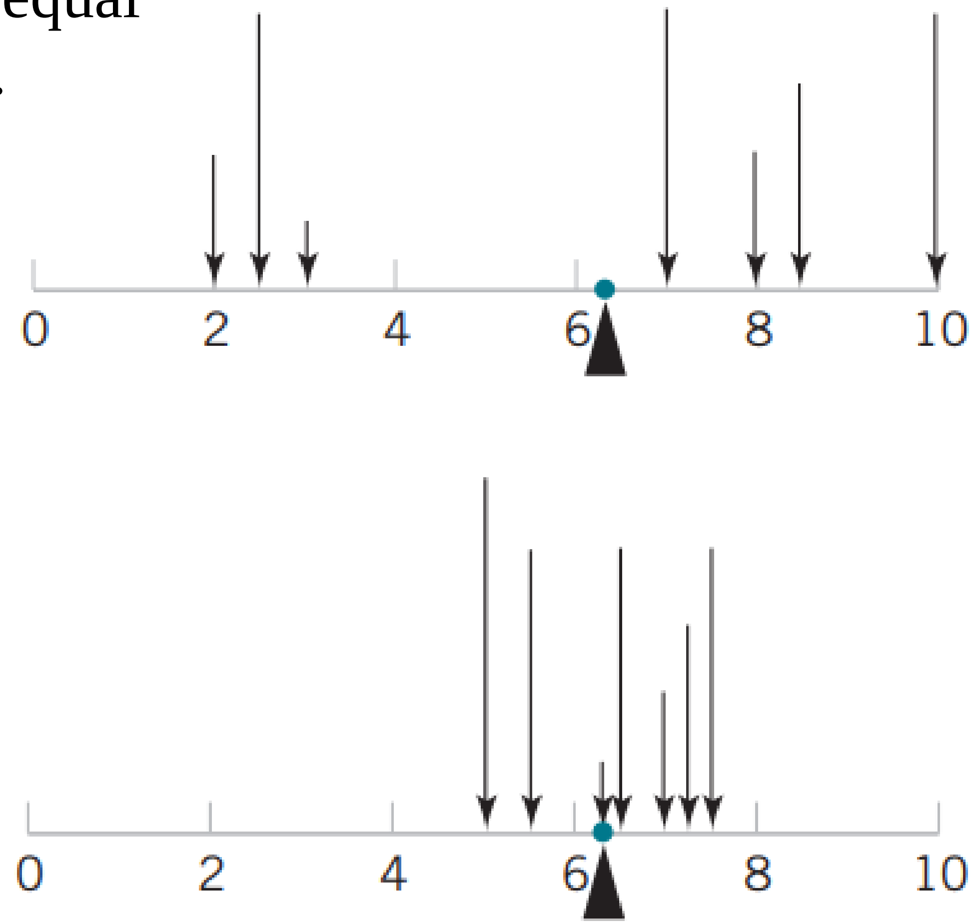
Discrete cumulative distribution function.

Mean and Variance (1/15)

Two numbers are often used to summarize a probability distribution for a random variable X. The mean is a measure of the center or middle of the probability distribution, and the variance is a measure of the dispersion, or variability in the distribution.

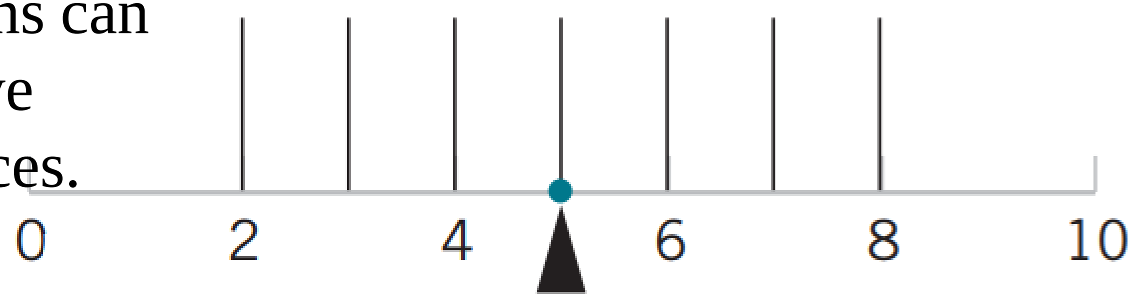
Mean and Variance (2/15)

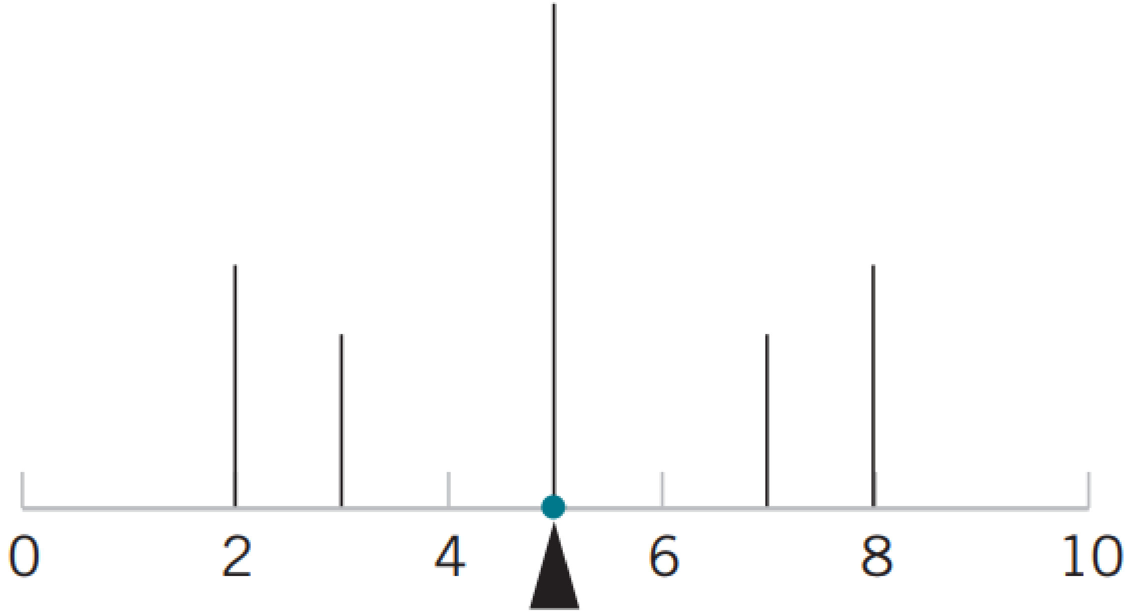
Probability distributions with equal means but different variances.



Mean and Variance (3/15)

Two probability distributions can differ even though they have identical means and variances.





Mean and Variance (4/15)

Mean, Variance, and Standard deviation

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x f(x)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

$$E(X^2) - \left(E(X)\right)^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance (5/15)

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

Determine the mean and variance of the random variable X

Mean and Variance (6/15)

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Find:

Determine the mean and variance of the random variable X

Answer: (1/2)

$E(X) =$

$$\sum x_i P(x_i) = (-2)\left(\frac{1}{8}\right) + (-1)\left(\frac{2}{8}\right) + (0)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (2)\left(\frac{1}{8}\right)$$

$$= 0$$

Mean and Variance (6/15)

Example1

x	-2	-1	0	1	2
$f(x) = P(X = x)$	1/8	2/8	2/8	2/8	1/8

Answer: (2/2)

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X) = 0$$

$$E(X^2)$$

$$= \sum x_i^2 P(x_i) = (4)\left(\frac{1}{8}\right) + (1)\left(\frac{2}{8}\right) + (0)\left(\frac{2}{8}\right) + (1)\left(\frac{2}{8}\right) + (4)\left(\frac{1}{8}\right) = 1.5$$

$V(X) = 1.5 - (0)^2 = 1.5,$ Standard Deviation $(\sigma) = \sqrt{1.5}$

Mean and Variance (7/15)

Example2:

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Mean and Variance (8/15)

Example2 – Answer (1/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Let X represent the number of good components in the sample. **Then x can only take the numbers 0, 1, 2 and 3.**

Example2 – Answer (2/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

The probability distribution of *X* is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Mean and Variance (8/15)

Example2 – Answer (3/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(0) = P(X = 0) = \frac{\binom{4}{0}\binom{3}{3}}{\binom{7}{3}} = \frac{1}{35}$$

Mean and Variance (8/15)

Example2 – Answer (4/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(1) = P(X = 1) = \frac{\binom{4}{1}\binom{3}{2}}{\binom{7}{3}} = \frac{12}{35}$$

Mean and Variance (8/15)

Example2 – Answer (5/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(2) = P(X = 2) = \frac{\binom{4}{2}\binom{3}{1}}{\binom{7}{3}} = \frac{18}{35}$$

Example2 – Answer (6/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

$$f(3) = P(X = 3) = \frac{\binom{4}{3}\binom{3}{0}}{\binom{7}{3}} = \frac{4}{35}$$

Mean and Variance (8/15)

Example2 – Answer (7/9)

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

Example2 – Answer (8/9)

Find the expected value of the number of good components in this sample.

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

$$E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Mean and Variance (8/15)

Example2 – Answer (9/9)

$E(X) = 1.7$

Determine the variance of the random variable X

x	0	1	2	3
$f(x) = P(X = x)$	1/35	12/35	18/35	4/35

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \sum x_i^2 P(x_i) = 0\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (4)\left(\frac{18}{35}\right) + (9)\left(\frac{4}{35}\right) = \frac{120}{35} = 3.43$$

$V(X) = 3.43 - (1.7)^2 = 0.54,$ Standard Deviation $(\sigma) = \sqrt{0.54} = 0.74$

Mean and Variance (9/15)

For any constants *a* and *b*:

Mean

1. $E(a) = a\,, \qquad a \in \mathbb{R}$
2. $E(aX + b) = aE(X) + b\,, \qquad a, b \in \mathbb{R}$

Variance

1. $V(a) = 0\,,\ \ a \in \mathbb{R}$
2. $V(aX + b) = a^2V(X)\,, \qquad a, b \in \mathbb{R}$

Mean and Variance (10/15)

Example3:

A discrete random variable with $V(X) = 2.5$
Evaluate $V(2X + 1)$

Mean and Variance (11/15)

Example3 – Answer

A discrete random variable with $V(X) = 2.5$
Evaluate $V(2X + 1)$

$$V(aX + b) = a^2V(X), \qquad a, b \in \mathbb{R}$$

$$V(2X + 1) = 4V(X) = 4 \times 2.5 = 10$$

Mean and Variance (12/15)

Example4:

A discrete random variable with $E(X) = 2.5$

Evaluate $E(2X + 1)$

Mean and Variance (13/15)

Example4 – Answer

A discrete random variable with $E(X) = 2.5$
Evaluate $E(2X + 1)$

$$E(aX + b) = aE(X) + b\,, \qquad a, b \in \mathbb{R}$$

$$E(2X + 1) = 2E(X) + 1$$

$$E(2X + 1) = 2 \times 2.5 + 1 = 6$$

Mean and Variance (14/15)

Example5:

Let X is a random variable with mean 6 and variance 100. Consider another random variable Y such that $Y = 3X + 6$, evaluate the mean and variance of Y?

Mean and Variance (15/15)

Example5 – Answer

Let X is a random variable with mean 6 and variance 100. Consider another random variable Y such that $Y = 3X + 6$, evaluate the mean and variance of Y?

$$E(X) = 6 \quad , \quad V(X) = 100$$

$$E(Y) = E(3X + 6)$$

$$V(Y) = V(3X + 6)$$

Mean and Variance (15/15)

Example5 – Answer

Let X is a random variable with mean 6 and variance 100. Consider another random variable Y such that $Y = 3X + 6$, evaluate the mean and variance of Y?

$$E(X) = 6 \quad , \quad V(X) = 100$$

$$E(Y) = E(3X + 6) = 3E(X) + 6 = 3(6) + 6 = 24$$

$$V(Y) = V(3X + 6) = 9V(X) = 9(100) = 900$$

Video Lectures

All Lectures: https://www.youtube.com/playlist?list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-

Lecture #4: https://www.youtube.com/watch?v=zWDzNUTfk9s&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=4 Start from 00:41:39

Notes Lec 1 – 4: https://www.youtube.com/watch?v=F9f61KpLeRk&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=5

https://www.youtube.com/watch?v=8X8D2ONdSK4&list=PLxIvc-MGOs6gW9SgkmoxE5w9vQkID1_r-&index=6 Until the 00:36:40

Thank You

Dr. Ahmed Hagag
ahagag@fci.bu.edu.eg